

L^p -Solutions of Reflected Backward Doubly Stochastic Differential Equations

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Abstract

In this paper, we deal with a class of one-dimensional reflected backward doubly stochastic differential equations with one continuous lower barrier. We derive the existence and uniqueness of L^p -solutions for those equations with Lipschitz coefficients.

Keywords: Reflected backward doubly stochastic differential equation; Lipschitz coefficient; L^p -solution

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1 Introduction

The general nonlinear case backward stochastic differential equation (BSDE in short) was first introduced in Pardoux and Peng (1990), who proved the existence and uniqueness result when the coefficient is Lipschitz. El Karoui et al. (1997a) introduced the notion of one barrier reflected BSDE, which is actually a backward equation but the solution is forced to stay above a given barrier. This type of BSDEs is motivated by pricing American options (see El Karoui et al. (1997b)) and studying the mixed game problems (see e.g. Cvitanić and Karatzas (1996), Hamadène and Lepeltier (2000)). In order

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to give a probabilistic representation for a class of quasilinear stochastic partial differential equations, Pardoux and Peng (1992) first considered a class of backward doubly stochastic differential equations (BDSDEs) with two different directions of stochastic integrals.

However in most of the previous works, solutions are taken in L^2 space or in $L^p, p > 2$. This limits the scope for several applications. To correct this shortcoming, El Karoui et al. (1997c) obtained the first result on the existence and uniqueness of solution in $L^p, p \in (1, 2)$ with a Lipschitz coefficient. Briand et al. (2003) generalized this result to the BSDEs with monotone coefficients. Following this way, Aman (2009) considered the L^p -solutions of BDSDEs with a monotone coefficient. Moreover, Hamadène and Popier (2008) established the existence and uniqueness of the L^p -solutions of BSDEs with reflection having a Lipschitz coefficient.

More recently, Bahalai et al. (2009) obtained the existence and uniqueness of solution for BDSDEs with one continuous lower barrier, having a continuous coefficient. Motivated by above works, the purpose of this paper is to prove the existence and uniqueness of L^p -solutions for reflected BDSDEs with Lipschitz coefficients.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries including some spaces. With the help of some a priori estimates, Section 3 is devoted to the existence and uniqueness of L^p -solutions for those equations.

2 Preliminaries

Let $T > 0$ a fixed real number. Let $\{W_t\}_{t \geq 0}, \{B_t\}_{t \geq 0}$ be two mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in \mathbf{R}^d and \mathbf{R} , respectively. For $t \in [0, T]$, we define

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where $\mathcal{F}_t^W = \sigma\{W_s, 0 \leq s \leq t\}$, $\mathcal{F}_{t,T}^B = \sigma\{B_s - B_t, t \leq s \leq T\}$ completed with the \mathbf{P} -null sets. We note that the collection $\{\mathcal{F}_t; t \in [0, T]\}$ is neither increasing nor decreasing, so it does not constitute a classical filtration. The Euclidean norm of a vector $y \in \mathbf{R}^n$ will be defined by $|y|$.

Throughout the paper, we always assume that $p \in (1, 2)$. Now, let's introduce the following spaces :

$\mathcal{M}_d^p = \{\psi : [0, T] \times \Omega \rightarrow \mathbf{R}^d, \text{predictable, such that } \mathbf{E}[(\int_0^T |\psi_s|^2 ds)^{\frac{p}{2}}] < \infty\};$

$\mathcal{S}^p = \{\psi : [0, T] \times \Omega \rightarrow \mathbf{R}, \text{ progressively measurable, s.t. } \mathbf{E}(\sup_{t \in [0, T]} |\psi_t|^p) < \infty\};$
 $\mathcal{S}_{ci}^p = \{A : [0, T] \times \Omega \rightarrow \mathbf{R}_+, \text{ continuous, increasing, s.t. } A_0 = 0 \text{ and } \mathbf{E}|A_T|^p < \infty\}.$

The object in this paper is the following reflected BDSDE:

$$\begin{cases} Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s \\ \quad + K_T - K_t - \int_t^T Z_s dW_s, \\ Y_t \geq L_t, 0 \leq t \leq T \text{ a.s. and } \int_0^T (Y_t - L_t)dK_t = 0, \text{ a.s.} \end{cases} \quad (1)$$

where the dW is a standard forward Itô integral and the dB is a backward Itô integral.

On the items ξ, f, g and L , we make the following assumptions:

(H1) The terminal condition $\xi : \Omega \rightarrow \mathbf{R}, \mathcal{F}_T$ -measurable such that $\mathbf{E}|\xi|^p < \infty$;

(H2) the functions $f, g : [0, T] \times \Omega \times \mathbf{R} \times \mathbf{R}^d \rightarrow \mathbf{R}$ are jointly measurable and satisfy:

(i) $\mathbf{E}[(\int_0^T |f_s^0|^2 ds)^{\frac{p}{2}}] < \infty, \mathbf{E}[(\int_0^T |g_s^0|^2 ds)^{\frac{p}{2}}] < \infty,$
where $f_s^0 =: f(s, 0, 0), g_s^0 =: g(s, 0, 0)$;

(ii) $\forall t \in [0, T], (y_1, z_1), (y_2, z_2) \in \mathbf{R} \times \mathbf{R}^d$, there exist constants $C > 0$ and $0 < \alpha < 1$ such that

$$\begin{aligned} |f(t, y_1, z_1) - f(t, y_2, z_2)| &\leq C(|y_1 - y_2| + |z_1 - z_2|), \\ |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 &\leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2; \end{aligned}$$

(H3) The barrier $\{L_t, t \in [0, T]\}$ is a real valued progressively measurable process such that $E(\sup_{0 \leq t \leq T} (L_t^+)^p) < \infty$ and $L_T \leq \xi$ a.s..

Let's give the notion of L^p -solution of reflected BDSDE (1).

Definition 2.1 *An L^p -solution of the reflected BDSDE (1) is a triple of progressively measurable processes (Y, Z, K) satisfying (1) such that $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}_d^p \times \mathcal{S}_{ci}^p$.*

The following lemma is a slight generalization of Corollary 2.3 in Briand et al. (2003).

Lemma 2.1 *Let $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}_d^p$ is a solution of the following BDSDE :*

$$|Y_t| = \xi + \int_t^T \tilde{f}(s, Y_s, Z_s)ds + \int_t^T \tilde{g}(s, Y_s, Z_s)dB_s + A_T - A_t - \int_t^T Z_s dW_s,$$

where:

- (i) \tilde{f} and \tilde{g} are functions which satisfy the assumptions as f and g ,
- (ii) \mathbf{P} -a.s. the process $(A_t)_{t \in [0, T]}$ is of bounded variation type.

Then for any $0 \leq t \leq u \leq T$, we have

$$\begin{aligned}
& |Y_t|^p + c(p) \int_t^T |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\
\leq & |Y_u|^p + p \int_t^T |Y_s|^{p-1} \widehat{Y}_s dA_s + p \int_t^T |Y_s|^{p-1} \tilde{f}(s, Y_s, Z_s) ds \\
& + c(p) \int_t^T |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |\tilde{g}(s, Y_s, Z_s)|^2 ds \\
& + p \int_t^T |Y_s|^{p-1} \widehat{Y}_s \tilde{g}(s, Y_s, Z_s) dB_s - p \int_t^T |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s,
\end{aligned}$$

where $c(p) = \frac{p(p-1)}{2}$ and $\widehat{y} = \frac{y}{|y|} \mathbf{1}_{\{y \neq 0\}}$.

3 Main results

3.1 A priori estimates

In order to obtain the existence and uniqueness result for solution of the reflected BDSDE (1), we first provide some a priori estimates of solution of (1).

In what follows, d, d_1, d_2, \dots will be denoted as a constant whose value depending only on C, α, p and possibly T . We also denote by $\theta_1, \theta_2, \dots$ the constants which taking value in $(0, \infty)$ arbitrarily.

Lemma 3.1 *Let the assumptions (H1)-(H3) hold and let (Y, Z, K) be a solution of the reflected BDSDE (1). If $Y \in \mathcal{S}^p$ then $Z \in \mathcal{M}_d^p$ and there exists a constant $d > 0$ such that*

$$\mathbf{E} \left[\left(\int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \leq d \mathbf{E} \left[\sup_{t \in [0, T]} |Y_t|^p + \left(\int_0^T |f_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^T |g_s^0|^2 ds \right)^{\frac{p}{2}} \right].$$

Proof. For each integer $n \geq 0$, let's define the stopping time

$$\tau_n = \inf \{ t \in [0, T], \int_0^t |Z_s|^2 ds \geq n \} \wedge T.$$

Let $a \in \mathbf{R}$, using Itô's formula and assumption (H2), we get

$$\begin{aligned}
& |Y_0|^2 + \int_0^{\tau_n} e^{as} |Z_s|^2 ds \\
&= e^{a\tau_n} |Y_{\tau_n}|^2 - a \int_0^{\tau_n} e^{as} |Y_s|^2 ds + 2 \int_0^{\tau_n} e^{as} Y_s f(s, Y_s, Z_s) ds \\
&\quad + \int_0^{\tau_n} e^{as} |g(s, Y_s, Z_s)|^2 ds + 2 \int_0^{\tau_n} e^{as} Y_s dK_s \\
&\quad + 2 \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s - 2 \int_0^{\tau_n} e^{as} Y_s Z_s dW_s \\
&\leq e^{a\tau_n} |Y_{\tau_n}|^2 - a \int_0^{\tau_n} e^{as} |Y_s|^2 ds \\
&\quad + \frac{1}{\theta_1} \int_0^{\tau_n} e^{as} |Y_s|^2 ds + \theta_1 \int_0^{\tau_n} e^{as} [4C^2(|Y_s|^2 + |Z_s|^2) + 2|f_s^0|^2] ds \\
&\quad + (1 + \theta_1) \int_0^{\tau_n} e^{as} (C|Y_s|^2 + \alpha|Z_s|^2) ds + (1 + \frac{1}{\theta_1}) \int_0^{\tau_n} e^{as} |g_s^0|^2 ds \\
&\quad + \frac{1}{\theta_2} \sup_{t \in [0, \tau_n]} e^{2at} |Y_t|^2 + \theta_2 |K_{\tau_n}|^2 \\
&\quad + 2 \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s - 2 \int_0^{\tau_n} e^{as} Y_s Z_s dW_s.
\end{aligned}$$

On the other hand, from the equation

$$K_{\tau_n} = Y_0 - Y_{\tau_n} - \int_0^{\tau_n} f(s, Y_s, Z_s) ds - \int_0^{\tau_n} g(s, Y_s, Z_s) dB_s + \int_0^{\tau_n} Z_s dW_s,$$

we have

$$\begin{aligned}
|K_{\tau_n}|^2 &\leq d_1 \left[|Y_0|^2 + |Y_{\tau_n}|^2 + \left(\int_0^{\tau_n} |f_s^0| ds \right)^2 + \int_0^{\tau_n} (|Y_s|^2 + |Z_s|^2) ds \right. \\
&\quad \left. + \left| \int_0^{\tau_n} g(s, Y_s, Z_s) dB_s \right|^2 + \left| \int_0^{\tau_n} Z_s dW_s \right|^2 \right].
\end{aligned}$$

Plugging this last inequality in the previous one to get

$$\begin{aligned}
& (1 - d_1\theta_2)|Y_0|^2 + (1 - 4C^2\theta_1 - \alpha(1 + \theta_1)) \int_0^{\tau_n} e^{as}|Z_s|^2 ds - d_1\theta_2 \int_0^{\tau_n} |Z_s|^2 ds \\
\leq & (d_1\theta_2 + e^{a\tau_n})|Y_{\tau_n}|^2 + \left(\frac{1}{\theta_1} + 4C^2\theta_1 + C(1 + \theta_1) - a \right) \int_0^{\tau_n} e^{as}|Y_s|^2 ds \\
& + 2\theta_1 \int_0^{\tau_n} e^{as}|f_s^0|^2 ds + (1 + \frac{1}{\theta_1}) \int_0^{\tau_n} e^{as}|g_s^0|^2 ds + \frac{1}{\theta_2} \sup_{t \in [0, \tau_n]} e^{2at}|Y_t|^2 \\
& + d_1\theta_2 \left[\int_0^{\tau_n} |f_s^0|^2 ds + \int_0^{\tau_n} |Y_s|^2 ds + \left| \int_0^{\tau_n} g(s, Y_s, Z_s) dB_s \right|^2 + \left| \int_0^{\tau_n} Z_s dW_s \right|^2 \right] \\
& + 2 \left| \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s \right| + 2 \left| \int_0^{\tau_n} e^{as} Y_s Z_s dW_s \right|.
\end{aligned}$$

Choosing now θ_1, θ_2 small enough and $a > 0$ such that $\frac{1}{\theta_1} + 4C^2\theta_1 + C(1 + \theta_1) - a < 0$, we obtain

$$\begin{aligned}
\int_0^{\tau_n} |Z_s|^2 ds \leq & d_2 \left(\sup_{t \in [0, \tau_n]} |Y_t|^2 + \int_0^{\tau_n} e^{as}|g_s^0|^2 ds + \int_0^{\tau_n} e^{as}|f_s^0|^2 ds \right. \\
& + \left| \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s \right| + \left| \int_0^{\tau_n} e^{as} Y_s Z_s dW_s \right| \\
& \left. + \theta_2 \left| \int_0^{\tau_n} g(s, Y_s, Z_s) dB_s \right|^2 + \theta_2 \left| \int_0^{\tau_n} Z_s dW_s \right|^2 \right), \quad (2)
\end{aligned}$$

it follows that

$$\begin{aligned}
& \mathbf{E} \left(\int_0^{\tau_n} |Z_s|^2 ds \right)^{\frac{p}{2}} \\
\leq & d_3 \mathbf{E} \left[\sup_{t \in [0, \tau_n]} |Y_t|^p + \left(\int_0^{\tau_n} |g_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^{\tau_n} |f_s^0|^2 ds \right)^{\frac{p}{2}} \right. \\
& + \left(\left| \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s \right| \right)^{\frac{p}{2}} + \left(\left| \int_0^{\tau_n} e^{as} Y_s Z_s dW_s \right| \right)^{\frac{p}{2}} \\
& \left. + \theta_2^{\frac{p}{2}} \left| \int_0^{\tau_n} g(s, Y_s, Z_s) dB_s \right|^p + \theta_2^{\frac{p}{2}} \left| \int_0^{\tau_n} Z_s dW_s \right|^p \right].
\end{aligned}$$

By the Burkholder-Davis-Gundy and Young's inequalities, we have

$$\begin{aligned}
& \mathbf{E} \left[\left| \int_0^{\tau_n} e^{as} Y_s g(s, Y_s, Z_s) dB_s \right|^{\frac{p}{2}} \right] \\
& \leq d_4 \mathbf{E} \left[\left(\int_0^{\tau_n} |Y_s|^2 |g(s, Y_s, Z_s)|^2 ds \right)^{\frac{p}{4}} \right] \\
& \leq d_4 \mathbf{E} \left[\left(\sup_{t \in [0, \tau_n]} |Y_t| \right)^{\frac{p}{2}} \left(\int_0^{\tau_n} |g(s, Y_s, Z_s)|^2 ds \right)^{\frac{p}{4}} \right] \\
& \leq \left(\frac{d_4}{\theta_3} + \theta_3 \right) \mathbf{E} \left[\sup_{t \in [0, \tau_n]} |Y_t|^p \right] \\
& \quad + \theta_3 \mathbf{E} \left[\left(\int_0^{\tau_n} |g_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^{\tau_n} |Z_s|^2 ds \right)^{\frac{p}{2}} \right]
\end{aligned}$$

and

$$\mathbf{E} \left[\left| \int_0^{\tau_n} e^{as} Y_s Z_s dW_s \right|^{\frac{p}{2}} \right] \leq \frac{d_5}{\theta_3} \mathbf{E} \left[\sup_{t \in [0, \tau_n]} |Y_t|^p \right] + \theta_3 \mathbf{E} \left[\left(\int_0^{\tau_n} |Z_s|^2 ds \right)^{\frac{p}{2}} \right].$$

Plugging the two last inequalities in the previous one and using the Burkholder-Davis-Gundy inequality once again, it follows after choosing θ_2 , θ_3 small enough (s.t. (2) holds too):

$$\begin{aligned}
& \mathbf{E} \left[\left(\int_0^{\tau_n} |Z_s|^2 ds \right)^{\frac{p}{2}} \right] \\
& \leq d \mathbf{E} \left[\sup_{t \in [0, \tau_n]} |Y_t|^p + \left(\int_0^{\tau_n} |f_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^{\tau_n} |g_s^0|^2 ds \right)^{\frac{p}{2}} \right].
\end{aligned}$$

Finally, we get the desired result by Fatou's Lemma. \square

Lemma 3.2 *Assume that (H1)-(H3) hold, let (Y, Z, K) be a solution of the reflected BDSDE (1) where $Y \in \mathcal{S}^p$. Then there exists a constant $d > 0$ such*

that

$$\begin{aligned}
& \mathbf{E} \left[\sup_{t \in [0, T]} |Y_t|^p + \left(\int_0^T |Z_s|^2 ds \right)^{\frac{p}{2}} + |K_T|^p \right] \\
& \leq d \mathbf{E} \left[|\xi|^p + \left(\int_0^T |f_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^T |g_s^0|^2 ds \right)^{\frac{p}{2}} \right. \\
& \quad \left. + \sup_{t \in [0, T]} (L_t^+)^p + \int_0^T |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right].
\end{aligned}$$

Proof. From Lemma 2.1, for any $a \in \mathbf{R}$ and any $0 \leq t \leq T$, we have

$$\begin{aligned}
& e^{apt} |Y_t|^p + c(p) \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\
& \leq e^{apT} |\xi|^p + p \int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s f(s, Y_s, Z_s) ds + p \int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s dK_s \\
& \quad + c(p) \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 ds - ap \int_t^T e^{aps} |Y_s|^p ds \\
& \quad + p \int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s g(s, Y_s, Z_s) dB_s - p \int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s. \quad (3)
\end{aligned}$$

By assumption (H2) and Young's inequality, we obtain

$$\begin{aligned}
& p \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s f(s, Y_s, Z_s) ds \right] \\
& \leq \mathbf{E} \left[p \int_t^T e^{aps} |Y_s|^{p-1} |f_s^0| ds + Cp \int_t^T e^{aps} |Y_s|^{p-1} (|Y_s| + |Z_s|) ds \right] \\
& \leq (p-1) \theta_4^{\frac{p}{p-1}} \mathbf{E} \left(\sup_{s \in [0, T]} |Y_s|^p \right) + \theta_4^{-p} \mathbf{E} \left(\int_t^T e^{aps} |f_s^0| ds \right)^p \\
& \quad + (Cp + \frac{p^2 C^2}{2c(p) \theta_4}) \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^p ds \right] \\
& \quad + \frac{c(p)}{2} \theta_4 \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right] \quad (4)
\end{aligned}$$

and

$$\begin{aligned}
& c(p) \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 ds \right] \\
\leq & c(p) \left(1 + \frac{1}{\theta_4}\right) \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right] \\
& + c(p) C(1 + \theta_4) \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^p ds \right] \\
& + c(p) \alpha(1 + \theta_4) \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right]. \tag{5}
\end{aligned}$$

Moreover, since $dK_s = \mathbf{1}_{\{Y_s \leq L_s\}} dK_s$, we get from Young's inequality

$$\begin{aligned}
& p \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s dK_s \right] \\
\leq & p \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s \mathbf{1}_{\{Y_s \leq L_s\}} dK_s \right] \\
\leq & p \mathbf{E} \left[\int_t^T e^{aps} |L_s|^{p-1} \widehat{L}_s dK_s \right] \\
\leq & p \mathbf{E} \left[\left(\sup_{s \in [0, T]} L_s^+ \right)^{p-1} \int_t^T e^{aps} dK_s \right] \\
\leq & \frac{p-1}{\theta_4^{\frac{p-1}{p}}} \mathbf{E} \left[\sup_{s \in [0, T]} (L_s^+)^p \right] + \theta_4^p \mathbf{E} \left(\int_t^T e^{aps} dK_s \right)^p \\
\leq & d_6 \left[\theta_4^{-\frac{p-1}{p}} \mathbf{E} \left(\sup_{s \in [0, T]} (L_s^+)^p \right) + \theta_4^p \mathbf{E} |K_T|^p \right].
\end{aligned}$$

On the other hand, by assumption (H2), the Burkholder-Davis-Gundy inequality and Lemma 3.1, we have

$$\mathbf{E} |K_T|^p \leq d_7 \mathbf{E} \left[\sup_{s \in [0, T]} |Y_s|^p + \left(\int_0^T |f_s^0| ds \right)^p + \left(\int_0^T |g_s^0| ds \right)^p \right], \tag{6}$$

it follows that

$$\begin{aligned}
p \mathbf{E} \left[\int_t^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s dK_s \right] & \leq d_8 \mathbf{E} \left[\theta_4^p \sup_{s \in [0, T]} |Y_s|^p + \theta_4^{-\frac{p-1}{p}} \sup_{s \in [0, T]} (L_s^+)^p \right. \\
& \quad \left. + \theta_4^p \left(\int_0^T |f_s^0| ds \right)^p + \theta_4^p \left(\int_0^T |g_s^0| ds \right)^p \right] \tag{7}
\end{aligned}$$

Combining (4)-(6), taking expectation on both sides of (3) to obtain

$$\begin{aligned}
& \mathbf{E} \left[e^{apt} |Y_t|^p + c(p) \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right] \\
\leq & \mathbf{E} \left\{ e^{apT} |\xi|^p + d_8 \theta_4^{-\frac{p-1}{p}} \sup_{s \in [0, T]} (L_s^+)^p + \left[(p-1) \theta_4^{\frac{p}{p-1}} + d_8 \theta_4^p \right] \sup_{s \in [0, T]} |Y_s|^p \right. \\
& + \left[(Cp + \frac{p^2 C^2}{2c(p)\theta_4}) + c(p)C(1 + \theta_4) - ap \right] \int_t^T e^{aps} |Y_s|^p ds \\
& + \left[\frac{c(p)}{2} \theta_4 + c(p)\alpha(1 + \theta_4) \right] \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |Z_s|^2 ds \\
& + \theta_4^{-p} \left(\int_t^T e^{aps} |f_s^0| ds \right)^p + d_8 \theta_4^p \left(\int_t^T |f_s^0| ds \right)^p + d_8 \theta_4^p \left(\int_t^T e^{aps} |g_s^0| ds \right)^p \\
& \left. + \left[c(p)(1 + \frac{1}{\theta_4}) + d_8 \theta_4^p \right] \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right\}.
\end{aligned}$$

Choosing θ_4 small enough and $a > 0$ such that

$$(Cp + \frac{p^2 C^2}{2c(p)\theta_4}) + c(p)C(1 + \theta_4) - ap < 0, \quad (8)$$

we get

$$\begin{aligned}
& \mathbf{E} \left[e^{apt} |Y_t|^p + c(p) \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right] \\
\leq & d_9 \mathbf{E} \left[|\xi|^p + \sup_{s \in [0, T]} (L_s^+)^p + \left(\int_t^T |f_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_t^T |g_s^0|^2 ds \right)^{\frac{p}{2}} \right. \\
& \left. + \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right] + d_9 \theta_4^p \mathbf{E} \left[\sup_{s \in [0, T]} |Y_s|^p \right]. \quad (9)
\end{aligned}$$

Next using the Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
& \mathbf{E} \left[\sup_{t \in [0, T]} \left| p \int_0^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s Z_s dW_s \right| \right] \\
\leq & \frac{1}{4} \mathbf{E} \left[\sup_{t \in [0, T]} e^{apt} |Y_t|^p \right] + d_{10} \mathbf{E} \left(\int_0^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right) \quad (10)
\end{aligned}$$

and

$$\begin{aligned}
& \mathbf{E} \left[\sup_{t \in [0, T]} \left| \int_0^T e^{aps} |Y_s|^{p-1} \widehat{Y}_s g(s, Y_s, Z_s) dB_s \right| \right] \\
& \leq \frac{1}{4} \mathbf{E} \left[\sup_{t \in [0, T]} e^{apt} |Y_t|^p \right] + d_{11} \mathbf{E} \int_0^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 ds \\
& \leq \frac{1}{4} \mathbf{E} \left[\sup_{t \in [0, T]} e^{apt} |Y_t|^p \right] + d_{11} \mathbf{E} \left[\int_0^T e^{aps} |Y_s|^{p-2} \mathbf{1}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right. \\
& \quad \left. + \int_0^T e^{aps} |Y_s|^p ds + \int_0^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right]. \tag{11}
\end{aligned}$$

Next going back to (3), using the Burkholder-Davis-Gundy inequality together with the inequalities (9)-(11), we get after choosing θ_4 small enough (s.t. inequality (8) holds too)

$$\begin{aligned}
& \mathbf{E} \left[\sup_{t \in [0, T]} e^{apt} |Y_t|^p + \int_t^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |Z_s|^2 ds \right] \\
& \leq d \mathbf{E} \left[|\xi|^p + \left(\int_0^T |f_s^0|^2 ds \right)^{\frac{p}{2}} + \left(\int_0^T |g_s^0|^2 ds \right)^{\frac{p}{2}} \right. \\
& \quad \left. + \sup_{t \in [0, T]} (L_t^+)^p + \int_0^T e^{aps} |Y_s|^{p-2} \mathbf{I}_{\{Y_s \neq 0\}} |g_s^0|^2 ds \right].
\end{aligned}$$

We then complete the proof by the inequality (6). \square

Lemma 3.3 *Let (Y', Z', K') and (Y, Z, K) be the solution of the reflected BDSDE (1) associated with (ξ', f', g', L) and (ξ, f, g, L) respectively, where (ξ', f', g', L) and (ξ, f, g, L) satisfy assumptions (H1)-(H3). Then*

$$\begin{aligned}
& \mathbf{E} \left[\sup_{t \in [0, T]} |Y'_t - Y_t|^p + \left(\int_0^T |Z'_s - Z_s|^2 ds \right)^{\frac{p}{2}} \right] \\
& \leq d \mathbf{E} \left[|\xi' - \xi|^p + \left(\int_0^T |f'(s, Y_s, Z_s) - f(s, Y_s, Z_s)|^2 ds \right)^{\frac{p}{2}} \right. \\
& \quad \left. + \left(\int_0^T |g'(s, Y_s, Z_s) - g(s, Y_s, Z_s)|^2 ds \right)^{\frac{p}{2}} \right].
\end{aligned}$$

Proof. The proof of the lemma is a combination of the proofs of Lemmas 3.1 and 3.2 with a slight change. Indeed, let

$$\bar{\xi} =: \xi' - \xi, \quad (\bar{Y}, \bar{Z}, \bar{K}) =: (Y' - Y, Z' - Z, K' - K).$$

One can easily check that $(\bar{Y}, \bar{Z}, \bar{K})$ is a solution to the following BDSDE:

$$\bar{Y}_t = \bar{\xi} + \int_t^T h(s, \bar{Y}_s, \bar{Z}_s) ds + \int_t^T k(s, \bar{Y}_s, \bar{Z}_s) dB_s + \bar{K}_T - \bar{K}_t - \int_t^T \bar{Z}_s dW_s,$$

where

$$\begin{aligned} h(s, y, z) &=: f'(s, y + Y_s, z + Z_s) - f(s, Y_s, Z_s), \\ k(s, y, z) &=: g'(s, y + Y_s, z + Z_s) - g(s, Y_s, Z_s). \end{aligned}$$

Obviously, the functions h and k are Lipschitz w.r.t (y, z) .

Let's note that

$$\int_0^t e^{aps} \bar{Y}_s d\bar{K}_s = - \int_0^t e^{aps} (Y'_s - L_s) dK_s - \int_0^t e^{aps} (Y_s - L_s) dK'_s \leq 0$$

and

$$\begin{aligned} \int_0^t e^{aps} |\bar{Y}_s|^{p-1} \widehat{\bar{Y}_s} d\bar{K}_s &= - \int_0^t e^{aps} |\bar{Y}_s|^{p-2} \mathbf{1}_{\{\bar{Y}_s \neq 0\}} (Y'_s - L_s) dK_s \\ &\quad - \int_0^t e^{aps} |\bar{Y}_s|^{p-2} \mathbf{I}_{\{\bar{Y}_s \neq 0\}} (Y_s - L_s) dK'_s \\ &\leq 0. \end{aligned}$$

The rest of the proof follows Itô's formula, Lemma 2.1 and the steps similar to those in the proofs of Lemmas 3.1 and 3.2. \square

3.2 Existence and uniqueness of a solution

In order to obtain the existence and uniqueness result, we make the following supplementary assumption:

(H4) $g(\cdot, 0, 0) \equiv 0$.

The following result due to Bahlali et al. (2009).

Lemma 3.4 *Let $p = 2$. Assume that (H1)-(H3) hold. Then the reflected BDSDE (1) has a unique solution $(Y, Z, K) \in \mathcal{S}^2 \times \mathcal{M}_d^2 \times \mathcal{S}_{ci}^2$.*

We now state and prove our main result.

Theorem 3.1 *Assume (H1)-(H4), then the reflected BDSDE (1) has a unique solution $(Y, Z, K) \in \mathcal{S}^p \times \mathcal{M}_d^p \times \mathcal{S}_{ci}^p$.*

Proof. The uniqueness is an immediate consequence of Lemma 3.3. We next to prove the existence.

For each $n, m \in \mathbb{N}^*$, define

$$\xi_n = q_n(\xi), f_n(t, x, y) = f(t, x, y) - f_t^0 + q_n(f_t^0), L_t^m = q_m(L_t),$$

where $q_k(x) = x \frac{k}{|x| \vee k}$. One can easily to check that the items ξ_n, f_n and L^m satisfy the assumptions (H1)-(H3), it follows from Lemma 3.4 that, for each $n, m \in \mathbb{N}^*$, there exists a unique solution $(Y^n, Z^n, K^n) \in L^2$ for the reflected BDSDE associated with (ξ_n, f_n, g, L^m) , but in fact also in L^p , according assumption (H4) and the Lemmas 3.1 and 3.2.

Next, from Lemma 3.3, for $(i, n) \in \mathbb{N} \times \mathbb{N}^*$, we have

$$\begin{aligned} & \mathbf{E} \left\{ \sup_{t \in [0, T]} |Y_t^{n+i} - Y_t^n|^p + \left(\int_0^T |Z_s^{n+i} - Z_s^n|^2 ds \right)^{\frac{p}{2}} \right\} \\ & \leq d \mathbf{E} \left\{ |\xi_{n+i} - \xi_n|^p + \left(\int_0^T |q_{n+i}(f_s^0) - q_n(f_s^0)|^2 ds \right)^{\frac{p}{2}} \right\}. \end{aligned}$$

Clearly, the right side of above inequality tend to 0 as $n \rightarrow \infty$, uniformly on i so that (Y^n, Z^n) is a Cauchy sequence in $\mathcal{S}^p \times \mathcal{M}_d^p$. Let's denote by $(Y, Z) \in \mathcal{S}^p \times \mathcal{M}_d^p$ it limit. By the equation

$$K_t^n = Y_0^n - Y_t^n - \int_0^t f_n(s, Y_s^n, Z_s^n) ds - \int_0^t g(s, Y_s^n, Z_s^n) dB_s + \int_0^t Z_s^n dW_s,$$

similar computation can derive that $(K_t^n)_{n \geq 1}$ is also a Cauchy sequence in \mathcal{S}_{ci}^p , then there exists a non-decreasing process $K_t \in \mathcal{S}_{ci}^p$ ($K_0 = 0$) such that

$$\mathbf{E}(|K_t^n - K_t|^p) \rightarrow 0, \text{ as } n \rightarrow \infty$$

and

$$\int_0^T (Y_s - L_s^m) dK_s = 0.$$

By the dominated convergence theorem, we then get

$$\int_0^T (Y_s - L_s^m) dK_s \rightarrow \int_0^T (Y_s - L_s) dK_s, \text{ as } m \rightarrow \infty.$$

It follows that the limit (Y, Z, K) is a L^p -solution of reflected BDSDE with (ξ, f, g, L) . The proof is complete. \square

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